Rotating perfect fluid sources of the NUT metric

Michael Bradley†, Gyula Fodor‡, László Á. Gergely‡¶, Mattias Marklund† and Zoltán Perjés‡

† Department of Plasma Physics, Umeå University, S–901 87 Umeå, Sweden
‡ KFKI Research Institute for Particle and Nuclear Physics, Budapest 114, P.O.Box 49, H–1525 Hungary
¶ Laboratoire de Physique Théorique, Université Louis Pasteur, 3–5 rue de l'Université, 67084 Strasbourg Cedex, France

Abstract. Locally rotationally symmetric perfect fluid solutions of Einstein's gravitational equations are matched along the hypersurface of vanishing pressure with the NUT metric. These rigidly rotating fluids are interpreted as sources for the vacuum exterior which consists only of a stationary region of the Taub-NUT space-time. The solution of the matching conditions leaves generally three parameters in the global solution. Examples of perfect fluid sources are discussed.

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1. Introduction

Matching fluids to vacuum exteriors has been a vexed question for many years in general relativity. Here we present a number of global solutions to Einstein's field equations representing rotating fluids matched to the vacuum NUT metric. The physical significance of the latter has also been in the center of discussion, with at least three differing interpretations in the literature [1, 2, 3]. The intepretation of Demianski and Newman has recently been revisited by Lynden–Bell and Nouri–Zonoz [4].

A recent paper by Mars and Senovilla [5] emphasizes the difficulty of matching an axisymmetric rotating interior body to a vacuum exterior. They show that the problem becomes overdetermined when both the continuity of the induced metric and of the extrinsic curvature of the junction hypersurface is imposed. Due to the overdetermination it is hard to find global space-times consisting of a rotating core and vacuum exterior. The weaker requirement of continuity of the metric alone was discussed by Shaud and Pfister [6].

The present paper was motivated by the structural similarity of some rotating perfect fluid solutions [7, 8, 9, 10] and the NUT metric [11]. We solve the junction conditions for the radius $r = r_s$ and the parameters of the NUT metric.

We cut the perfect fluid space-time along the hypersurface of zero pressure, this being an obvious requirement for matching with a vacuum solution. In fact, the junction conditions single out the zero pressure surface as the unique matching surface. Then

we apply the Darmois–Israel matching procedure [12, 13]. The continuity of both the first and second fundamental forms impose relations among the parameters of the two solutions.

The fluid space-times are locally rotationally symmetric (LRS), i.e., they are invariant under a subgroup of the Lorentz group. Further, the LRS class under study has non-zero vorticity but zero shear and expansion of the fluid. The general system of equations determining the properties of the fluid has been presented in many forms [7, 9, 14].

In section 2, we briefly review the the theory of LRS perfect fluid space-times. We also state the properties of the NUT space-time. Next, in section 3 we derive the general junction conditions, and finally in section 4 we discuss some examples of the matching procedure and the matching space-times.

Our metric signature convention is $(+---)$, and we choose the gravitational constant $G = (8\pi)^{-1}$.

2. Properties of the LRS perfect fluid and the NUT space-times

The locally rotationally symmetric (LRS) space-times are defined to be invariant under a spatial isotropy subgroup of the Lorentz group. This means that the subgroup is either 1- or 3-dimensional. The LRS space-times may be divided into three distinct classes [7, 14, 15]. LRS models were also studied by Cahen and Defrise [9]. In this paper we will make use of class I according to the Stewart–Ellis classification [14]. In the generic case, this class is invariant under a 1-dimensional isotropy group.

We choose the following Lorentz tetrad:

$$\begin{aligned}
\omega^0 &= \frac{\Omega\delta^2}{L}(\mathrm{d}T + 2L\cos\Theta\,\mathrm{d}\Phi) \\
\omega^1 &= \frac{\mathrm{d}R}{y} \\
\omega^2 &= \delta\,\mathrm{d}\Theta \\
\omega^3 &= \delta\sin\Theta\,\mathrm{d}\Phi\ .
\end{aligned} \tag{1}$$

where L is a constant and

$$\delta^{-2} \equiv -p + 2a\kappa + \kappa^2 - \Omega^2\ . \tag{2}$$

Here p is the pressure and, denoting the Ricci rotation coefficients by γ_{ijk},

$$\Omega = -\gamma_{023}\ , \quad \kappa = -\gamma_{122} = -\gamma_{133}\ , \quad a = -\gamma_{010} \tag{3}$$

are the vorticity, the spatial divergence of the basis vector e_1 dual to ω^1, and the acceleration, respectively.‡ The function y is determined by the choice of the radial coordinate R.

‡ The Ricci rotation coefficients γ_{ijk} satisfy Cartan's first equation $\mathrm{d}\omega^i = \gamma^i{}_{jk}\omega^j \wedge \omega^k$.

The functions in the tetrad depends only on the R coordinate, and they are subject to the radial equations

$$
\begin{aligned}
y\frac{d\Omega}{dR} &= (2\kappa - a)\Omega \\
y\frac{d\kappa}{dR} &= \tfrac{1}{2}(\mu + p) - a\kappa - \Omega^2 + \kappa^2 \\
y\frac{da}{dR} &= -\tfrac{1}{2}(\mu + 3p) + a^2 + 2a\kappa + 2\Omega^2 \\
y\frac{dp}{dR} &= a(\mu + p)
\end{aligned}
\tag{4}
$$

where μ is the energy-density of the fluid.

The static limit is characterized by $\Omega = 0$ (note that in this case the tetrad (1) can not be used). This subclass includes all static spherically symmetric perfect fluid space-times [16], e.g., the Tolman [17] and interior Schwarzschild solutions.

If we choose our coordinate R such that $\delta^2 = R^2 + L^2$, then from (2) and (4) $y = -\kappa(R^2 + L^2)/R$. Since p is determined algebraically from equation (2), we may discard the fourth of the above radial equations.

The metric (1) bears a striking similarity to that of the stationary region of the Taub–NUT vacuum space-time [2, 11, 18], which in a Lorentz frame takes the form

$$
\begin{aligned}
\tilde{\omega}^0 &= h(\mathrm{d}t + 2\ell\cos\theta\,\mathrm{d}\varphi) \\
\tilde{\omega}^1 &= \frac{\mathrm{d}r}{h} \\
\tilde{\omega}^2 &= (r^2 + \ell^2)^{1/2}\,\mathrm{d}\theta \\
\tilde{\omega}^3 &= (r^2 + \ell^2)^{1/2}\sin\theta\,\mathrm{d}\varphi
\end{aligned}
\tag{5}
$$

where

$$
h^2 = \frac{r^2 - 2mr - \ell^2}{r^2 + \ell^2} \tag{6}
$$

and m and ℓ are constants. Using Cartan's first equation, we obtain the Ricci rotation coefficients, among which

$$
\begin{aligned}
\tilde{\gamma}_{010} &= \frac{mr^2 - m\ell^2 + 2r\ell^2}{(r^2 + \ell^2)^{3/2}(r^2 - \ell^2 - 2mr)^{1/2}} \\
\tilde{\gamma}_{122} = \tilde{\gamma}_{133} &= \frac{r(r^2 - \ell^2 - 2mr)^{1/2}}{(r^2 + \ell^2)^{3/2}}\ .
\end{aligned}
\tag{7}
$$

are needed for matching.

3. The junction conditions between the LRS and NUT space-times

The general junction conditions can be stated as follows: Find two isometric imbeddings of the matching surface $\mathcal{S}$ (with metric $\mathrm{d}s^2_{\mathcal{S}}$) into the given "interior" and "exterior"

space-times respectively, such that the induced extrinsic curvatures of $\mathcal{S}$ may be equated with each other. Thus, the following should hold on the matching 3-surface:

$$K_{ij} = \tilde{K}_{ij} \ , \quad \mathrm{d}s^2|_{\mathcal{S}} = \mathrm{d}\tilde{s}^2|_{\mathcal{S}} \ . \tag{8}$$

If the surface $\mathcal{S}$ is time-like, choosing a normal n to the surface yields the extrinsic curvature $K_{ij} = h_i^k h_j^l n_{(k;l)}$ (and analogously for $\tilde{K}$), where $h_{ij} = g_{ij} + n_i n_j$ is the projector onto $\mathcal{S}$.

We choose the surface $\mathcal{S}$ such that ω^1 and $\tilde{\omega}^1$ are normals to the imbedded surface in the two space-time regions. Thus, we take $n = \omega^1$, i.e., $n_i = \delta_i^1$. This implies that $\mathcal{S}$ is a surface of constant radius r or R.

The non-trivial tetrad components of the extrinsic curvature are given as the Ricci rotation coefficients:

$$K_{\alpha\beta} = \gamma_{1(\alpha\beta)} \ , \tag{9}$$

where $\alpha, \beta = 0, 2, 3$. Then the matching conditions are

$$\gamma_{1(\alpha\beta)} = \tilde{\gamma}_{1(\alpha\beta)} \tag{10}$$

$$\omega^\alpha|_{\mathcal{S}} = \tilde{\omega}^\alpha|_{\mathcal{S}} \ . \tag{11}$$

Making the coordinate choice $R^2 = \delta^2 - L^2$ in the LRS case, and using equations (1), (3), (5), and (7), we find

$$\ell^2 = \frac{\Omega_s^2 r_s^2}{\kappa_s^2} \tag{12}$$

$$\kappa_s = -\frac{r_s(r_s^2 - \ell^2 - 2mr_s)^{1/2}}{(r_s^2 + \ell^2)^{3/2}} \tag{13}$$

$$a_s = -\frac{mr_s^2 - m\ell^2 + 2r_s\ell^2}{(r_s^2 + \ell^2)^{3/2}(r_s^2 - \ell^2 - 2mr_s)^{1/2}} \ , \tag{14}$$

with $R_s = r_s$ and $L = \ell$, and where the subscript s is used to denote quantities on $\mathcal{S}$. We choose $t = T$, $\theta = \Theta$, and $\varphi = \Phi$ on the junction surface $\mathcal{S}$. Using the above given radial coordinate definition, equation (2) is found to be trivially satisfied on the zero-pressure surface due to the junction conditions. Therefore it may be used in place of, say, equation (14). We solve equations (12), (13) and (2) for the NUT parameters and the matching radius, and obtain

$$\ell^2 = \frac{\Omega_s^2}{(\kappa_s^2 + \Omega_s^2)(2a_s\kappa_s + \kappa_s^2 - \Omega_s^2)} \tag{15}$$

$$mr_s = \frac{\kappa_s(a_s\kappa_s^2 - a_s\Omega_s^2 - 2\kappa_s\Omega_s^2)}{(\kappa_s^2 + \Omega_s^2)(2a_s\kappa_s + \kappa_s^2 - \Omega_s^2)^2} \tag{16}$$

$$r_s^2 = \frac{\kappa_s^2}{(\kappa_s^2 + \Omega_s^2)(2a_s\kappa_s + \kappa_s^2 - \Omega_s^2)} \tag{17}$$

with the subsidiary condition $2a_s\kappa_s + \kappa_s^2 - \Omega_s^2 > 0$ (and $mr_s > 0$, if we want a positive mass parameter). In order to confine the metric to the stationary domain of the Taub–NUT space-time, then the matching radius r_s should be larger than the limiting radius

of the Taub region $r_{\rm T} = m + \sqrt{m^2 + \ell^2}$, a condition expressed as $r_s^2 - \ell^2 - 2mr_s > 0$. This is satisfied, since

$$r_s^2 - \ell^2 - 2mr_s = \frac{\kappa_s^2 + \Omega_s^2}{(2a_s\kappa_s + \kappa_s^2 - \Omega_s^2)^2} \ . \tag{18}$$

The static limit ($\Omega = 0$) of the fluid requires $\ell = 0$. One hence gets a matching to the exterior Schwarzschild solution in this limit.

In the generic case, having chosen an equation of state $\mu = \mu(p)$, the integral of equations (4) contains three constants after the coordinate R has been chosen. Therefore there remain three independent parameters in the matching conditions. Finding the solution in the generic case remains difficult even with the equation of state specified. Simplifying assumptions yield metrics with a reduced number of parameters. This will imply that the NUT parameters and the junction radius are not independent parameters. In certain cases, with examples given in the next section, it will not be possible to satisfy all the positivity conditions.

4. Explicit global solutions

When joining the two metrics, the radial coordinate of the fluid has to be specified. The coordinate choice $R^2 = \delta^2 - L^2$ is convenient for our matching procedure, but it covers only the region outside the surface $\delta = L$. Thus, in the fluid interior, it is more appropriate to use the function δ as the radial coordinate. In each case presented below, the metric may be reconstructed using the tetrads (1) and (5).

4.1. Incompressible fluid with constant vorticity

Assuming that the vorticity $\Omega \equiv \Omega_s$ and the density $\mu \equiv \mu_s$ are constants, equations (4) yield $a = 2\kappa$. Furthermore, the density, pressure and acceleration are given in the form [7, 8]

$$\mu_s = 6\Omega_s^2 \ , \quad p = \frac{4}{\delta^2} - 6\Omega_s^2 \ , \quad a^2 = \frac{4}{\delta^2} - 4\Omega_s^2 \ . \tag{19}$$

The coordinate choice $\delta^2 = R^2 + L^2$ gives the junction conditions (15)–(17) in the simple form

$$\ell^2 = \frac{4}{9\Omega_s^2} \ , \quad mr_s = -\frac{4}{9\Omega_s^2} \ , \quad r_s^2 = \frac{2}{9\Omega_s^2} \ . \tag{20}$$

There is only one free parameter, Ω_s. We see further that the mass is negative. It is straightforward to show that this defect can not be removed by including a cosmological constant.

4.2. A fluid with variable density and constant vorticity

As in the previous case, this fluid has $\Omega \equiv \Omega_s$ a constant and $a = 2\kappa$, but with an equation of state

$$p = 12\Omega_s^2 - 3\mu + A(\mu - 6\Omega_s^2)^{3/2} \ , \tag{21}$$

with $\mu > 6\Omega_s^2$ guaranteeing the positivity of the speed of sound. Here A is a constant. The density, pressure and the acceleration may be expressed as [7]

$$\mu = \left(\frac{4}{A\delta^2}\right)^{2/3} + 6\Omega_s^2 \tag{22}$$

$$p = \frac{4}{\delta^2} - 3\left(\frac{4}{A\delta^2}\right)^{2/3} - 6\Omega_s^2 \tag{23}$$

$$a^2 = \frac{4}{\delta^2} - \frac{12}{5}\left(\frac{4}{A\delta^2}\right)^{2/3} - 4\Omega_s^2 \; . \tag{24}$$

On the junction surface $\mathcal{S}$, with the coordinate choices as before, we have the matching conditions

$$\ell^2 = \frac{5\Omega_s^2\delta_s^4}{1+6\Omega_s^2\delta_s^2} \tag{25}$$

$$mr_s = \frac{2}{5}\frac{\delta_s^2(1+\Omega_s^2\delta_s^2)(1-19\Omega_s^2\delta_s^2)}{1+6\Omega_s^2\delta_s^2} \tag{26}$$

$$r_s^2 = \delta_s^2\frac{1+\Omega_s^2\delta_s^2}{1+6\Omega_s^2\delta_s^2} \tag{27}$$

where

$$f = 18\left(\sqrt{\tfrac{2}{3}+(A\Omega_s)^2} - A\Omega_s\right) \tag{28}$$

$$\delta_s^2 = \frac{f^{1/3}}{3A\Omega_s^2} - \frac{2}{A\Omega_s^3 f^{1/3}} + \frac{2}{3\Omega_s^2} \; . \tag{29}$$

The positivity of the mass m is guaranteed by the inequality

$$|A\Omega_s| < \left(\frac{2}{19}\right)^{1/2}\left(\frac{57}{35}\right)^{3/2} \approx 0.67 \; . \tag{30}$$

4.3. Purely electric Weyl tensor

The condition that the Weyl tensor is purely electric [7, 10] yields that $a = \kappa$ and

$$p = -\frac{1}{2}\mu_s + \frac{1}{8D-2}\delta^{-2} \tag{31}$$

$$\Omega^2 = \frac{1-2D}{8D-2}\delta^{-2} \tag{32}$$

$$a^2 = -\frac{1}{6}\mu_s + \frac{2D}{8D-2}\delta^{-2} \; , \tag{33}$$

where μ_s is a positive constant. The equation of state is given by

$$\mu = p + \mu_s \; . \tag{34}$$

The reality of these quantities implies $1/4 < D < 1/2$.

Using the junction conditions (15)–(17), we find the following expressions for the NUT parameters and the junction radius

$$\ell^2 = \frac{3(1-2D)}{8\mu_s(1-4D)(1-3D)} \tag{35}$$

$$mr_s = \frac{(468D^2 - 288D + 43)(12D-5)}{24\mu_s(11-30D)(1-4D)^2(1-3D)} \tag{36}$$

$$r_s^2 = \frac{30D-11}{8\mu_s(1-4D)(1-3D)} \ . \tag{37}$$

The positivity of the mass parameter forces the value of D to lie in the open interval $(5/12, 1/2)$. Note that there are two free parameters, D and μ_s, in the above junction relations.

5. Discussion

In the previous section we have obtained global solutions to Einstein's equations, with the interiors representing a rigidly rotating perfect fluid each with a given equation of state. The matching with the exterior NUT space-time is along a freely chosen surface, selected by the appropriate choice of the parameters such that the pressure vanishes on this surface.

Because of the structure of the metric, the fluid interior shares the counter-intuitive properties of the exterior NUT space-time, e.g., closed time-like curves. In general, the LRS class I fluids all have curvature singularities at the center $\delta = 0$ (there may exist solutions which can not be extended to $\delta = 0$, although their physical status is quite unclear).

We emphasize that these fluids do not have a unique axis of rotation, instead they appear to rotate about every radial direction.

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